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Dynamic portfolio choice and asset pricing with differential information

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Abstract

This paper presents a multi-asset intertemporal general equilibrium model of portfolio selection and asset pricing with differential information. A method of Sargent (1991) is used to resolve the 'infinite regress' problem in information extraction and to derive a rational expectations equilibrium. The model shows that rational investors trade stocks strategically according to their perceptions about economic states and provides a rationale for investors to hold less than perfectly diversified portfolios. The information distribution among investors has an important effect on stock prices, welfare, and the investment opportunities of investors. The model helps explain a number of interesting financial regularities such as imperfect portfolio diversification and home bias. Published by Elsevier Science B.V.

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1. Introduction

Portfolio choice and asset pricing under heterogeneous information in a multi-asset securities market are interesting and challenging issues in modern finance. Admati (1985) addresses this issue in a static setup. Zhou (1997) builds a dynamic model to study the issue under a special information structure:

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information sets are completely ranked. However, the information structure in reality is more general and much richer. While some traders are better informed about certain aspects of the securities market, other traders may have better knowledge about some other aspects of the market. In other words, different investors may have different information which cannot be completely ranked. In line with He and Wang (1995), we use the term *differential information* to represent the information structure where heterogeneous information sets cannot be completely ranked.

This paper considers an interesting example of a differential information story. In a noisy two-stock market, there are two classes of traders. (Of course, one can extend it to any number of classes and any number of stocks in a straightforward way.) Class *a* traders are computer engineers who have better information and experience about the computer industry, especially IBM corporation; class *b* traders are communication experts who have better knowledge and insights about the communication industry, especially AT&T. In a frictionless market, should class *a* traders only hold IBM stock and class *b* traders only hold AT&T shares; or should they just hold the market portfolio? What is the difference between *a* and *b*'s portfolios? What are the effects of interaction between traders on stock prices? The answers to these questions are useful for explaining a broad range of phenomena in the empirical literature, including mean reversion, excess volatility, and especially, home bias in international portfolio choices.[1]

In solving a dynamic rational-expectations asset pricing model with differential information, one often faces the so-called 'infinite regress' problem regarding rational information extraction of economic agents (i.e., 'forecasting the forecasts of forecasts $\cdots$' of others). In this paper, we use an apparatus of Marcet and Sargent 1989a,b) and Sargent (1991) to handle it. Instead of modeling the beliefs of each class of economic agents as unobserved state variables, economic agents are modeled as forecasting the future by fitting finite-dimensional vector ARMA models for all information available to them, including endogenous variables such as prices.

Other work which is closely related to this paper includes He and Wang (1995) and Hussman (1992). He and Wang present a differential information model with a finite horizon and an infinite number of investors, while Hussman gives a model with two classes of traders in which each class observes a component of stock dividends. Both models assume a single risky asset. The current work can be viewed as an extension of these previous papers in two ways. First, it presents a multi-asset model which can explore the cross-sectional properties

[1] See, e.g., Cooper and Kaplanis (1994), French and Poterba (1991), Stulz (1994), and Tesar and Werner (1993) for evidence and discussion.

of asset prices. Second, it has implications for a number of real world financial issues such as imperfect portfolio diversification and the 'home-bias puzzle' which the previous papers do not have.

The rest of this paper is structured as follows. Section 2 describes the economic model. Section 3 considers the benchmark case of a perfect information discrete-time model. Section 4 shows rational information extraction in a noisy market with differential information sets which cannot be completely ranked. Section 5 solves for a differential information equilibrium of the market. Section 6 uses a couple of numerical examples to show economic implications of the current models and to explain some important findings cited in the empirical literature. Section 7 concludes.

2. The economic model

In this paper, we will consider a hypothetical exchange economy where one riskless asset and more than one risky asset are traded. Economic agents are differently informed, but no one informationally dominates all other agents. Formally, we have the following assumptions:

Assumption 1 (*Physical good*). There is only a single physical good in the economy, which can be allocated either to consumption or to investment. All values are expressed in the units of this good.

Assumption 2 (*Equity*). This is a multi-asset economy. For simplicity and without loss of generality, we assume that there are two risky assets (stock 1: IBM stock and stock 2: AT&T stock) available in the economy. The dividend process for each stock is driven by a (partially) persistent component and a (purely) transitory component

$$D_{it} = F_{it} + v_{iD,t}, \tag{1}$$

$$F_{it} = a_{iF}F_{i,t-1} + v_{iF,t}, \quad (-1 \leq a_{iF} \leq 1), \tag{2}$$

where D_{it} is the dividend payment of stock i in period t, F_{it} is the persistent component of D_{it} and $v_{iD,t}$ is the transitory component of D_{it}. Noise terms $v_{iD,t}$ and $v_{iF,t}$ are i.i.d. Gaussian processes with means zero and variances σ_{iD}^2 and σ_{iF}^2, respectively.

Assumption 3 (*Bond*). There is one risk-free asset (bond) which generates a fixed rate of dividend $r (r > 0)$ per unit time. The bond supply is perfectly elastic, so the price of bond will not be affected by the bond demand.

Assumption 4 (*Equity supply*). The total supply of each stock i ($i = 1,2$) is normalized to $1 + N_i$, where N_i is the noisy supply of stock i. N_i follows an AR(1) process:

$$N_{it} = a_{iN}N_{i,t-1} + v_{iN,t} \quad (-1 \leq a_{iN} \leq 1), \tag{3}$$

where $v_{iN,t}$ is an i.i.d Gaussian process with mean zero and variance σ_{iN}^2. We will call the total supply of a stock with noise the *noisy supply* of the stock and the total supply of a stock excluding noise the *pure supply* of the stock and will call the market portfolio with noise components the *noisy market portfolio* and the market portfolio excluding noise components the *pure market portfolio*.

Assumption 5 (*Information structure*). There are two classes of rational economic agents, indexed by $j = a,b$. The total population is normalized to 1, with a proportion k in class a and $1 - k$ in class b. Class a has perfect information about F_1 but does not observe F_2. Symmetrically, class b has perfect information about F_2 but does not observe F_1 of stock 1. Nobody observes noisy asset supplies. Mathematically, their information sets can be represented by

$$\mathscr{F}_t^a = \{P_{1\tau}, P_{2\tau}, D_{1\tau}, D_{2\tau}, F_{1\tau} | \tau \leq t\}, \tag{4}$$

$$\mathscr{F}_t^b = \{P_{1\tau}, P_{2\tau}, D_{1\tau}, D_{2\tau}, F_{2\tau} | \tau \leq t\}. \tag{5}$$

For expositional convenience, we sometimes simply call a representative agent of class a agent a and a representative agent of class b agent b.

Assumption 6 (*Common knowledge*). The structure of the economy is common knowledge.

Assumption 7 (*Preferences*). All economic agents have the same constant absolute risk aversion (CARA) preference. At any time t, agents maximize their expected utilities of next period wealth W_{t+1} by solving

$$\max \mathrm{E}_t[u(W_{t+1})] = \max \mathrm{E}_t[-\exp(-\phi W_{t+1})], \quad \phi > 0. \tag{6}$$

Assumption 8 (*Trading mechanism*). Trading in assets takes place once each period t at equilibrium prices P_{1t} and P_{2t} after dividends for that period D_{1t} and D_{2t} have been paid out. No trading takes place at non-equilibrium prices.

For simplicity, we assume the following covariance relations: $\mathrm{Cov}(v_D, v_F) = \mathrm{Cov}(v_D, v_N) = \mathrm{Cov}(v_F, v_N) = 0$, $\mathrm{Cov}(v_{1D}, v_{2D}) = \eta_D$, $\mathrm{Cov}(v_{1F}, v_{2F}) = \eta_F$, and

$\mathrm{Cov}(v_{1N},v_{2N}) = \eta_N$. That is, we assume that the shocks in different categories are uncorrelated but that the shocks in the same categories can be correlated. Every random shock is assumed to be i.i.d over time.

A comment on our notation is in order here. We use letters with subscript i ($i = 1,2$) to denote coefficients or variables associated with stock i, e.g., a_{iF} and F_i. After dropping subscript i, those letters in boldface will represent the corresponding diagonal matrices (for coefficients) or column vectors (for variables), e.g.,

$$\boldsymbol{a}_F = \begin{bmatrix} a_{1F} & 0 \\ 0 & a_{2F} \end{bmatrix} \quad \text{and} \quad \boldsymbol{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \tag{7}$$

We use the capital Greek letter $\boldsymbol{\Sigma}$ (with subscripts) to represent variance-covariance matrices, e.g., $\boldsymbol{\Sigma}_F = \mathrm{Var}(\boldsymbol{F}) = \mathrm{E}[\boldsymbol{v}_F\boldsymbol{v}_F']$.

Generally, variables used in this paper have a time subscript while constants do not have a time subscript. When no confusion exists, subscripts may be suppressed.

3. Benchmark case: perfect information equilibrium

Before proceeding to study the differential information model, we will first consider the perfect information equilibrium in which rational economic agents observe the current and the past values of all of the underlying economic variables described earlier. This relatively simple perfect information setup provides useful intuition and will serve as a benchmark for evaluating the differential information equilibrium considered in subsequent sections.

3.1. Stock fundamentals and investment opportunities

We define the fundamental value of a stock as the expected present value of its dividend flows discounted at the riskless interest rate r. It is easy to see that

Theorem 1. *The fundamental value $V_i(t)$ of stock i is given by*

$$V_i(t) = \Theta_i^* F_i(t), \quad i = 1,2, \tag{8}$$

where $\Theta_i^* = a_{iF}/(1 + r - a_{iF})$.

Proof. By Assumption 2

$$V_{it} \equiv \mathrm{E}_t\left[\sum_{s=1}^{\infty} \frac{1}{(1+r)^s} D_{i,t+s}\right] \tag{9}$$

$$= \sum_{s=1}^{\infty} \left(\frac{a_{iF}}{1+r}\right)^s F_{it} \tag{10}$$

$$= \frac{a_{iF}}{1+r-a_{iF}} F_{it} \quad \square \tag{11}$$

To obtain the market equilibrium, we need to describe the investment opportunities first. Let Π_{it} denote the undiscounted cumulative cash flow from a zero-wealth portfolio long one share of stock i financed by selling the risk-free bond. We have

$$\Pi_{i,t+1} \equiv (P_{i,t+1} + D_{i,t+1}) - (1+r)P_{it} \tag{12}$$

$$= e_{i\Pi} + v_{i\Pi}, \tag{13}$$

where P_i is the price of stock i, $e_{i\Pi} = \mathrm{E}_t[\Pi_{i,t+1}]$ is the one-period-ahead expectation of excess return and $v_{i\Pi} = \Pi_i - e_{i\Pi}$ is the corresponding expectation error.

3.2. The equilibrium

According to Assumption 7, a representative economic agent's optimization problem can be written as

$$\max_{Q_t} \mathrm{E}_t[-\exp(-W_{t+1})], \tag{14}$$

subject to

$$W_{t+1} = (1+r)W_t + \boldsymbol{Q}_t'\boldsymbol{e}_{\Pi,t} + \boldsymbol{Q}_t'\boldsymbol{v}_{\Pi,t+1}, \tag{15}$$

where W is the agent's wealth and $\boldsymbol{Q}$ is the vector of his or her stock holdings.

Let us conjecture that $\boldsymbol{v}_\Pi$ is Gaussian.[2] With the conjecture, we immediately have that

$$\boldsymbol{Q} = \boldsymbol{\Sigma}_\Pi^{-1}\boldsymbol{e}_\Pi = \boldsymbol{\Sigma}_\Pi^{-1}\mathrm{E}_t[\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1} - (1+r)\boldsymbol{P}_t], \tag{16}$$

where $\boldsymbol{\Sigma}_\Pi$ is the variance–covariance matrix of the innovations $\boldsymbol{v}_\Pi$.

[2] We will see shortly that the conjecture is true since Eq. (18) implies that $\boldsymbol{P}_t$ and therefore $\boldsymbol{\Pi}_t$ are linear functions of $\boldsymbol{D}_t$, $\boldsymbol{F}_t$ and $\boldsymbol{N}_t$.

The total supply of stocks to rational economic agents is $\mathbf{1} + \boldsymbol{N}$, where $\mathbf{1}$ is a two-dimensional vector of ones. Market clearing condition $\boldsymbol{Q} = \mathbf{1} + \boldsymbol{N}$ then implies

$$\boldsymbol{P}_t = (1 + r)^{-1}\mathrm{E}_t(\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1}) - (1 + r)^{-1}\boldsymbol{\Sigma}_\Pi(\mathbf{1} + \boldsymbol{N}_t) \tag{17}$$

which may be solved forward to yield

$$\boldsymbol{P}_t = \boldsymbol{V}_t - (1/r)\boldsymbol{\Sigma}_\Pi\mathbf{1} - \boldsymbol{\Sigma}_\Pi\boldsymbol{\Phi}\boldsymbol{N}_t, \tag{18}$$

where $\boldsymbol{\Phi}$ is a 2×2 diagonal matrix

$$\boldsymbol{\Phi} = \begin{bmatrix} \dfrac{1}{1 + r - a_{1N}} & 0 \\ 0 & \dfrac{1}{1 + r - a_{2N}} \end{bmatrix}. \tag{19}$$

Theorem 2. The equilibrium conditions of the model imply that

$$\boldsymbol{e}_\Pi = \boldsymbol{\Sigma}_\Pi(\mathbf{1} + \boldsymbol{N}) \tag{20}$$

and that $\boldsymbol{\Sigma}_\Pi$ satisfies the following matrix equation:

$$\boldsymbol{\Sigma}_\Pi - \boldsymbol{\Sigma}_\Pi\boldsymbol{\Phi}\boldsymbol{\Sigma}_N\boldsymbol{\Phi}'\boldsymbol{\Sigma}_\Pi = \boldsymbol{\Sigma}_D + \boldsymbol{\Psi}\boldsymbol{\Sigma}_F\boldsymbol{\Psi}', \tag{21}$$

where $\boldsymbol{\Psi} = \boldsymbol{I} + \boldsymbol{\Theta}^*$ is a 2×2 matrix.

Proof. The first part of the theorem about $\boldsymbol{e}_\Pi$ is pretty straightforward since market clearing implies $\boldsymbol{Q} = \mathbf{1} + \boldsymbol{N}$.

Note that $\boldsymbol{D}_{t+1} = \boldsymbol{F}_{t+1} + \boldsymbol{v}_{D,t+1}$ as specified in Section 2. From price equation, Eq. (18), we have

$$\begin{aligned} \boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1} &= \boldsymbol{V}_{t+1} + \boldsymbol{D}_{t+1} - (1/r)\boldsymbol{\Sigma}_\Pi 1 - \boldsymbol{\Sigma}_\Pi\boldsymbol{\Phi}\boldsymbol{N}_{t+1} \\ &= \boldsymbol{\Theta}^*\boldsymbol{F}_{t+1} + \boldsymbol{F}_{t+1} + \boldsymbol{v}_{D,t+1} - (1/r)\boldsymbol{\Sigma}_\Pi\mathbf{1} - \boldsymbol{\Sigma}_\Pi\boldsymbol{\Phi}\boldsymbol{N}_{t+1} \\ &= \boldsymbol{\Psi}\boldsymbol{F}_{t+1} + \boldsymbol{v}_{D,t+1} - (1/r)\boldsymbol{\Sigma}_\Pi\mathbf{1} - \boldsymbol{\Sigma}_\Pi\boldsymbol{\Phi}\boldsymbol{N}_{t+1}, \end{aligned} \tag{22}$$

where $\boldsymbol{\Psi} = \boldsymbol{I} + \boldsymbol{\Theta}^*$.

On the other hand, the definition of $\boldsymbol{\Pi}$ implies that

$$\boldsymbol{\Sigma}_\Pi = \mathrm{Var}_t(\boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1}). \tag{23}$$

As a result, we have

$$\boldsymbol{\Sigma}_\Pi = \boldsymbol{\Sigma}_D + \boldsymbol{\Psi}\boldsymbol{\Sigma}_F\boldsymbol{\Psi}' + \boldsymbol{\Sigma}_\Pi\boldsymbol{\Phi}\boldsymbol{\Sigma}_N\boldsymbol{\Phi}'\boldsymbol{\Sigma}_\Pi. \quad \square \tag{24}$$

Eq. (21) has a real-valued solution if and only if the matrix $\boldsymbol{\Phi}\boldsymbol{\Sigma}_N\boldsymbol{\Phi}'$ is not too large in magnitude. Therefore, if the market is too noisy and/or the noise is too persistent, no stable equilibrium can be established. We will exclude this possibility in the subsequent analysis.

4. Information filtration and perceived investment opportunities

4.1. Perceived laws of motion

Now we consider the differential information model. To solve for an equilibrium with non-completely-ranked information sets, we need a tractable method to deal with the information extraction problem.

According to Assumption 5, agents in class a observe a record of current and past values

$$\boldsymbol{S}_{at} = [\tilde{P}_{1t}, \tilde{P}_{2t}, D_{1t}, D_{2t}, F_{1t}]', \tag{25}$$

where $\tilde{P}_{1t} = P_{1t} - p_1$ and $\tilde{P}_{2t} = P_{2t} - p_2$ are 'demeaned' stock prices. p_1 and p_2 reflect the unconditional expected risk premia, which will be discussed later.

Define $\boldsymbol{X}_{at} = \boldsymbol{S}_{at} - \mathrm{E}[\boldsymbol{S}_{at}|\mathscr{F}^a_{t-1}]$ as the period-ahead conditional expectation error in $\boldsymbol{S}_{at}$. Following Sargent (1991), we assume that the *filtration rule* of agent a, or equivalently, the agent's *perceived law of motion* for $\boldsymbol{S}_{at}$, is a first-order ARMA process of the form

$$\boldsymbol{S}_{at} = \boldsymbol{A}_a \boldsymbol{S}_{a,t-1} + \boldsymbol{B}_a \boldsymbol{X}_{a,t-1} + \boldsymbol{X}_{at}. \tag{26}$$

We will solve for matrices $\boldsymbol{A}_a$ and $\boldsymbol{B}_a$ and show that this assumption is appropriate to establish a rational expectations equilibrium.

The above perceived law of motion can also be written as

$$\begin{bmatrix} \boldsymbol{S}_{at} \\ \boldsymbol{X}_{at} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_a & \boldsymbol{B}_a \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{S}_{a,t-1} \\ \boldsymbol{X}_{a,t-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{X}_{at} \\ \boldsymbol{X}_{at} \end{bmatrix}, \tag{27}$$

or

$$\boldsymbol{Y}_{at} = \boldsymbol{g}_a \boldsymbol{Y}_{a,t-1} + \boldsymbol{v}_{Ya,t}, \tag{28}$$

where

$$\boldsymbol{Y}_{at} = \begin{bmatrix} \boldsymbol{S}_{at} \\ \boldsymbol{X}_{at} \end{bmatrix}, \quad \boldsymbol{v}_{Ya,t} = \begin{bmatrix} \boldsymbol{X}_{at} \\ \boldsymbol{X}_{at} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{g}_a = \begin{bmatrix} \boldsymbol{A}_a & \boldsymbol{B}_a \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}. \tag{29}$$

$\boldsymbol{S}_{bt}$, $\boldsymbol{X}_{bt}$, $\boldsymbol{Y}_{bt}$ and $\boldsymbol{g}_b$ can be defined and analyzed symmetrically for agents in class b. For example, $\boldsymbol{S}_{bt}$ is defined as

$$\boldsymbol{S}_{bt} = [\tilde{P}_{1t}, \tilde{P}_{2t}, D_{1t}, D_{2t}, F_{2t}] \tag{30}$$

and then $\boldsymbol{X}_{bt} = \boldsymbol{S}_{bt} - \mathrm{E}[\boldsymbol{S}_{bt}|\mathscr{F}^b_{t-1}]$ is defined straightforwardly.

Given the perceptions outlined above, agents form period-ahead forecasts according to

$$\mathrm{E}[\boldsymbol{Y}_{at}|\boldsymbol{Y}_{a,t-1}] = \boldsymbol{g}_a\boldsymbol{Y}_{a,t-1}, \tag{31}$$

$$\mathrm{E}[\boldsymbol{Y}_{bt}|\boldsymbol{Y}_{b,t-1}] = \boldsymbol{g}_b\boldsymbol{Y}_{b,t-1}. \tag{32}$$

The actual law of motion for prices results from the dynamic market equilibrium that equates asset supplies and asset demands arising from these expectations. The rational expectations assumption requires that agents' perceptions be consistent with the actual law of motion.

Let z_t denote the state vector of the economy which contains $\boldsymbol{Y}_{at}$, $\boldsymbol{Y}_{bt}$ and some other state variables. With a proper choice of elements, z_t will evolve according to

$$z_t = \boldsymbol{T}(\boldsymbol{g})z_{t-1} + \boldsymbol{H}(\boldsymbol{g})\boldsymbol{w}_t, \tag{33}$$

where $\boldsymbol{w}_t$ is a vector of innovations. For a given set of perceptions $\boldsymbol{g} = (\boldsymbol{g}_a, \boldsymbol{g}_b)$, the actual law of motion, Eq. (33), can be used to obtain the projections of $\boldsymbol{Y}_{jt}$ on $\boldsymbol{Y}_{j,t-1}$ for $j = a,b$.

$$\mathrm{E}[\boldsymbol{Y}_{at}|\boldsymbol{Y}_{a,t-1}] = \boldsymbol{\Gamma}_a(\boldsymbol{g})\boldsymbol{Y}_{a,t-1}, \tag{34}$$

$$\mathrm{E}[\boldsymbol{Y}_{bt}|\boldsymbol{Y}_{b,t-1}] = \boldsymbol{\Gamma}_b(\boldsymbol{g})\boldsymbol{Y}_{b,t-1}, \tag{35}$$

where $\boldsymbol{\Gamma}_j(\boldsymbol{g})(j = a, b)$ are obtained using the linear least squares projection formula.

For the current asset pricing model, state vector z_t can be expressed as

$$z_t = \{\tilde{P}_{1t}, \tilde{P}_{2t}, D_{1t}, D_{2t}, F_{1t}, F_{2t}, N_{1t}, N_{2t}, \boldsymbol{X}_{at}, \boldsymbol{X}_{bt}\}' \tag{36}$$

and the vector of innovations in z_t, $\boldsymbol{w}_t$, can be written as

$$\boldsymbol{w}_t = [v_{1D}, v_{2D}, v_{1F}, v_{2F}, v_{1N}, v_{2N}]' \tag{37}$$

The equilibrium of the market can be formally defined as:

Definition. A (limited-information) rational expectations equilibrium (REE) with heterogeneous information is the fixed point $(\boldsymbol{g}_a, \boldsymbol{g}_b) = (\boldsymbol{\Gamma}_a(\boldsymbol{g}), \boldsymbol{\Gamma}_b(\boldsymbol{g}))$ such that the market clears in equilibrium.

This kind of equilibrium concept was previously used by Sargent (1991) in investigating optimal investment in a production economy, and was then used by Hussman (1992) in an asset pricing model similar to ours.[3] Both authors have

[3] Hussman (1992) assumes that there is a single risky asset in the market. As we mentioned in the introduction, this single-asset setup is not appropriate to address the effects of private information on portfolio choices and related issues.

discussed the properties of this equilibrium concept in detail, so we will not discuss them further. Below we use this equilibrium concept to investigate various implications of our multi-asset differential information asset pricing model.

4.2. Investment opportunities

Now we consider the optimization problems faced by rational economic agents given the perceived laws of motion. Because of the symmetry between classes a and b, we will only consider a's optimization behavior.

Investment opportunities characterize the distributions of stock returns. Denote $\boldsymbol{\Pi}_{t+1} = \boldsymbol{P}_{t+1} + \boldsymbol{D}_{t+1} - (\mathbf{1} + \boldsymbol{r})\boldsymbol{P}_t$ as the excess returns earned by each share of stock. Then based on agent a's information sets, $\boldsymbol{\Pi}$ can be expressed as

$$\begin{aligned}\boldsymbol{\Pi}_{t+1} &= -r\boldsymbol{p} + \boldsymbol{h}\boldsymbol{g}_a\boldsymbol{Y}_{at} - (1+r)\tilde{\boldsymbol{P}}_t + \boldsymbol{h}\boldsymbol{v}_{Ya,t+1} \\ &= -r\boldsymbol{p} + \boldsymbol{h}\boldsymbol{g}_a\boldsymbol{Y}_{at} - (1+r)\tilde{\boldsymbol{h}}\boldsymbol{Y}_{at} + \boldsymbol{h}\boldsymbol{v}_{Ya,t+1}, \\ &= \boldsymbol{e}_{\Pi a,t} + \boldsymbol{v}_{\Pi a,t+1},\end{aligned} \tag{38}$$

where

$$\boldsymbol{h} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{39}$$

$$\tilde{\boldsymbol{h}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{40}$$

$$\begin{aligned}\boldsymbol{e}_{\Pi at} &= -r\boldsymbol{p} + \boldsymbol{h}\boldsymbol{g}_a\boldsymbol{Y}_{at} - (1+r)\tilde{\boldsymbol{P}}_t \\ &= -r\boldsymbol{p} + \boldsymbol{h}\boldsymbol{g}_a\boldsymbol{Y}_{at} - (1+r)\tilde{\boldsymbol{h}}\boldsymbol{Y}_{at},\end{aligned} \tag{41}$$

$$\boldsymbol{v}_{\Pi a} = \boldsymbol{h}\boldsymbol{v}_{Ya}. \tag{42}$$

$\boldsymbol{h}$ and $\tilde{\boldsymbol{h}}$ are selector matrices.

Given investment opportunities, for a portfolio $\boldsymbol{Q}_a$, agent a receives a total excess payoff $\boldsymbol{Q}_a'\boldsymbol{\Pi}$. His wealth therefore evolves according to

$$\begin{aligned}W_{a,t+1} &= (1+r)W_{at} + \boldsymbol{Q}_a'\boldsymbol{\Pi} \\ &= (1+r)W_{at} + \boldsymbol{Q}_a'\boldsymbol{e}_{\Pi a} + \boldsymbol{Q}_a'\boldsymbol{v}_{\Pi a},\end{aligned} \tag{43}$$

where $W_{a,t+1}$ is agent a's wealth at time $t+1$.

$\boldsymbol{e}_{\Pi b}$, $\boldsymbol{v}_{\Pi b}$ and W_b for agent b can be analyzed symmetrically. Although $\boldsymbol{\Pi}$ is the same for everybody in the stock market, different people may have different perceptions of it. This difference will lead to different portfolio demands between agents and therefore to interesting trading dynamics.

5. Differential information equilibrium

Using the information extraction method introduced above, we now solve for a market equilibrium under differential information.

5.1. Trading strategy and portfolio choice

Let W_a be agent a's wealth and $\boldsymbol{Q}_a$ be the vector of agent a's stockholdings. Similar to the corresponding perfect information model, the optimization problem is

$$\max_{\boldsymbol{Q}_a} \mathrm{E}_t[-\exp(-W_{a,t+1})], \tag{44}$$

subject to

$$W_{a,t+1} = (1+r)W_{at} + \boldsymbol{Q}'_{at}\boldsymbol{e}_{\Pi a,t} + \boldsymbol{Q}'_{at}\boldsymbol{v}_{\Pi a,t+1}, \tag{45}$$

where all symbols are defined as before.

It follows directly that the optimal stock portfolio of agent a is

$$\begin{aligned}\boldsymbol{Q}_a &= \boldsymbol{\Sigma}_{\Pi a}^{-1}\boldsymbol{e}_{\Pi a}\\ &= \boldsymbol{\Sigma}_{\Pi a}^{-1}[-r\boldsymbol{p} + \boldsymbol{h}\boldsymbol{g}_a\boldsymbol{Y}_a - (1+r)\tilde{\boldsymbol{P}}],\end{aligned} \tag{46}$$

where $\boldsymbol{\Sigma}_{\Pi,a} = \mathrm{E}_t[\boldsymbol{v}_{\Pi a,t+1}\boldsymbol{v}'_{\Pi a,t+1}]$.

One can solve agent b's optimization problem symmetrically and gets

$$\begin{aligned}\boldsymbol{Q}_b &= \boldsymbol{\Sigma}_{\Pi,b}^{-1}\boldsymbol{e}_{\Pi,b},\\ &= \boldsymbol{\Sigma}_{\Pi b}^{-1}[-r\boldsymbol{p} + \boldsymbol{h}\boldsymbol{g}_b\boldsymbol{Y}_b - (1+r)\tilde{\boldsymbol{P}}],\end{aligned} \tag{47}$$

where $\boldsymbol{Q}_b$ is agent b's stock portfolio and $\boldsymbol{\Sigma}_{\Pi,b} = \mathrm{E}_t[\boldsymbol{v}_{\Pi b,t+1}\boldsymbol{v}'_{\Pi b,t+1}]$ is the variance–covariance matrix of $\boldsymbol{v}_{\Pi b}$.

The portfolio demand equations, Eqs. (46) and (47), show that rational economic agents partially diversify their portfolios in light of their private information. For example, agent a holds both stocks even though he or she is better informed about stock 1 and less informed about stock 2. Agent a's demand for each stock is proportional to the expectation of excess return about that stock and inversely proportional to the perceived risk of that stock. Hence an agent who is a computer expert holds not only IBM shares but also AT&T shares. In fact, the demand equation tells us that sometimes more AT&T shares may be held by agent *a*. (For example, agent *a* observes that the fundamental value of IBM has dropped sharply while other agents do not observe it.) The next section will illustrate numerically the portfolio demands of differently informed agents.

Information heterogeneity causes risk heterogeneity of investment opportunities. As we know from portfolio demand equations, Eqs. (46) and (47), the variance–covariance matrices $\boldsymbol{\Sigma}_{\Pi a}$ and $\boldsymbol{\Sigma}_{\Pi b}$ play a critical role in determining the difference between agent *a*'s portfolios and agent *b*'s portfolios. As we know, for a random variable, the more precise the information is, the smaller is the conditional variance of that variable. Since agent *a* has better information on Π_1 while agent *b* has more precise information about Π_2, $\sigma^2_{1\Pi,a} < \sigma^2_{1\Pi,b}$ and $\sigma^2_{2\Pi,a} > \sigma^2_{2\Pi,b}$, where $\sigma^2_{i\Pi,j}$ is the variance of stock *i*'s excess payoff conditional on agent *j*'s information. Intuitively, we can expect from this property that *on average* agent *a* puts more weight on stock 1 and agent *b* puts more weight on stock 2. The magnitude of the difference depends on to what extent endogenous variables (stock prices) convey the private information of one class to the other class. This is a rational information extraction problem. We will solve it numerically in Section 6.

5.2. Market clearing

Market clearing requires that the aggregate demand for each stock equal its aggregate supply. Thus

$$k\boldsymbol{Q}_a + (1-k)\boldsymbol{Q}_b = \mathbf{1} + \boldsymbol{N}. \tag{48}$$

From the portfolio demand equations, Eqs. (46) and (47), we know

$$\mathbf{1} = -[k\boldsymbol{\Sigma}_{\Pi a}^{-1} + (1-k)\boldsymbol{\Sigma}_{\Pi b}^{-1}](r\boldsymbol{p}) \tag{49}$$

$$\begin{aligned}\boldsymbol{N} = {} & k\boldsymbol{\Sigma}_{\Pi a}^{-1}[\boldsymbol{h}\boldsymbol{g}_a\boldsymbol{Y}_a - (1+r)\tilde{\boldsymbol{P}}] \\ & + (1-k)\boldsymbol{\Sigma}_{\Pi b}^{-1}[\boldsymbol{h}\boldsymbol{g}_b\boldsymbol{Y}_b - (1+r)\tilde{\boldsymbol{P}}].\end{aligned} \tag{50}$$

The market clearing condition, Eq. (50), together with perceived laws for $\boldsymbol{Y}_{at}$ and $\boldsymbol{Y}_{bt}$, gives

$$\tilde{\boldsymbol{P}}_t = \frac{1}{1+r}[k\boldsymbol{\Sigma}_{\Pi a}^{-1} + (1-k)\boldsymbol{\Sigma}_{\Pi b}^{-1}]^{-1}[k\boldsymbol{\Sigma}_{\Pi a}^{-1}\boldsymbol{h}\boldsymbol{g}_a\boldsymbol{d}_a + (1-k)\boldsymbol{\Sigma}_{\Pi b}^{-1}\boldsymbol{h}\boldsymbol{g}_b\boldsymbol{d}_b - \boldsymbol{d}_N]\boldsymbol{z}_t, \tag{51}$$

where $\boldsymbol{d}_a$, $\boldsymbol{d}_b$ and $\boldsymbol{d}_N$ are selector matrices such that

$$\boldsymbol{Y}_{at} = \boldsymbol{d}_a \boldsymbol{z}_t, \tag{52}$$

$$\boldsymbol{Y}_{bt} = \boldsymbol{d}_b \boldsymbol{z}_t, \tag{53}$$

$$\boldsymbol{N}_t = \boldsymbol{d}_N \boldsymbol{z}_t. \tag{54}$$

We will use this relation to derive the perceived laws $\boldsymbol{g}_a$ and $\boldsymbol{g}_b$.

5.3. Solving for the equilibrium

This subsection follows Sargent (1991) and Hussman (1992) to pin down agents' perceived laws of state motion, $\boldsymbol{g}_a$ and $\boldsymbol{g}_b$. For this purpose, we first have to figure out $\boldsymbol{T}(\boldsymbol{g})$ and $\boldsymbol{H}(\boldsymbol{g})$ in Eq. (33), where $\boldsymbol{g} = [\boldsymbol{g}_a', \boldsymbol{g}_b']'$.

Rows 3–8 of $\boldsymbol{T}$ and $\boldsymbol{H}$, corresponding to state variables D_1, D_2, F_1, F_2, N_1, and N_2 respectively, are implied by Eqs. (1)–(3) in the model specification.

Define selector matrices $\boldsymbol{e}_j$ and $\boldsymbol{f}_j$ such as

$$\boldsymbol{S}_{jt} = \boldsymbol{e}_j \boldsymbol{z}_t, \tag{55}$$

$$\boldsymbol{X}_{jt} = \boldsymbol{f}_j \boldsymbol{z}_t, \quad j = a,b. \tag{56}$$

Then from Eq. (26) we know

$$\boldsymbol{X}_{jt} = [\boldsymbol{e}_j \boldsymbol{T}(\boldsymbol{g}) - \boldsymbol{A}_j \boldsymbol{e}_j - \boldsymbol{B}_j \boldsymbol{f}_j] \boldsymbol{z}_{t-1} + \boldsymbol{e}_j \boldsymbol{H}(\boldsymbol{g}) \boldsymbol{w}_t. \tag{57}$$

Since the first rows of $\boldsymbol{T}$ and $\boldsymbol{H}$ are already given, and $\boldsymbol{e}_a$ does not select from the last rows of $\boldsymbol{T}$ and $\boldsymbol{H}$ corresponding to $\boldsymbol{X}_a$ and $\boldsymbol{X}_b$, the rows of $\boldsymbol{T}$ and $\boldsymbol{H}$ corresponding to $\boldsymbol{X}_a$ and $\boldsymbol{X}_b$ can be completely determined by the above equation.

Now we go back to consider how to determine the first two rows of $\boldsymbol{T}$ and $\boldsymbol{H}$: $\boldsymbol{T}_P$ and $\boldsymbol{H}_P$. Substituting Eq. (33) into Eq. (51), we get

$$\begin{aligned}
\tilde{\boldsymbol{P}}_t = {} & \frac{1}{1+r}[k\boldsymbol{\Sigma}_{\Pi a}^{-1} + (1-k)\boldsymbol{\Sigma}_{\Pi b}^{-1}]^{-1} \\
& \times [k\boldsymbol{\Sigma}_{\Pi a}^{-1}\boldsymbol{h}\boldsymbol{g}_a\boldsymbol{d}_a + (1-k)\boldsymbol{\Sigma}_{\Pi b}^{-1}\boldsymbol{h}\boldsymbol{g}_b\boldsymbol{d}_b - \boldsymbol{d}_N]\boldsymbol{T}(\boldsymbol{g})\boldsymbol{z}_{t-1} \\
& + \frac{1}{1+r}[k\boldsymbol{\Sigma}_{\Pi a}^{-1} + (1-k)\boldsymbol{\Sigma}_{\Pi b}^{-1}]^{-1} \\
& \times [k\boldsymbol{\Sigma}_{\Pi a}^{-1}\boldsymbol{h}\boldsymbol{g}_a\boldsymbol{d}_a + (1-k)\boldsymbol{\Sigma}_{\Pi b}^{-1}\boldsymbol{h}\boldsymbol{g}_b\boldsymbol{d}_b - \boldsymbol{d}_N]\boldsymbol{H}(\boldsymbol{g})\boldsymbol{w}_t.
\end{aligned} \tag{58}$$

Assume that the eigenvalues of matrix T are all inside the unit circle. This allows the computation of the stationary covariance matrix of z_t. Eq. (33) implies that the covariance matrix $\boldsymbol{\Sigma}_z = \mathrm{E}[z_t z_t']$ satisfies the following discrete Lyapunov equation:

$$\boldsymbol{\Sigma}_z = \boldsymbol{T}(\boldsymbol{g})\boldsymbol{\Sigma}_z\boldsymbol{T}(\boldsymbol{g})' + \boldsymbol{H}(\boldsymbol{g})\boldsymbol{\Sigma}_w\boldsymbol{H}(\boldsymbol{g})', \tag{59}$$

where $\boldsymbol{\Sigma}_w = \mathrm{E}[\boldsymbol{w}_t\boldsymbol{w}_t']$ is the covariance matrix of innovations $\boldsymbol{w}_t$.

Given the covariance matrix $\boldsymbol{\Sigma}_z$, the selector vectors $\boldsymbol{s}_j$ may be defined so that

$$\boldsymbol{\Sigma}_{Xj} \equiv \mathrm{E}[\boldsymbol{X}_{jt}\boldsymbol{X}_{jt}'] = \boldsymbol{s}_j\boldsymbol{\Sigma}_z\boldsymbol{s}_j', \quad j = a,\, b. \tag{60}$$

The optimal projection laws $\boldsymbol{\Gamma}_j(g)$ in Eqs. (31) and (32) are given by

$$\mathrm{E}[\boldsymbol{Y}_{jt}|\boldsymbol{Y}_{j,t-1}] = \boldsymbol{\Gamma}_j(\boldsymbol{g})\boldsymbol{Y}_{j,t-1}, \tag{61}$$

where

$$\boldsymbol{\Gamma}_j(\boldsymbol{g}) = \boldsymbol{d}_j\boldsymbol{T}(\boldsymbol{g})\boldsymbol{\Sigma}_z\boldsymbol{d}_j'[\boldsymbol{d}_j\boldsymbol{\Sigma}_z\boldsymbol{d}_j']^{-1}, \tag{62}$$

where $\boldsymbol{d}_j$ are the selector matrices such that $\boldsymbol{Y}_{jt} = \boldsymbol{d}_j\boldsymbol{z}_t$ defined as before.

In some cases, the matrix $[\boldsymbol{d}_j\boldsymbol{\Sigma}_z\boldsymbol{d}_j']$ may become singular, due to linear dependence in the observables of a or b. This problem may be circumvented, following Sargent (1991), by choosing matrices $\boldsymbol{d}_j$ to restrict the set of regressors used to compute $\boldsymbol{\Gamma}_j(\boldsymbol{g})$. The columns in $\boldsymbol{A}_j$, $\boldsymbol{B}_j$ corresponding to the excluded regressors are assigned zero values. Using the resulting equilibrium, we can straightforwardly calculate the coefficient of determination in the regression of the excluded regressors onto included regressors. If this coefficient is unity, the restriction does not constrain the information sets of a and b in equilibrium. In our example, for class a agents, since $D_1 = F_1 + v_{1D}$, we exclude F_1 from the regressors used to compute $\boldsymbol{\Gamma}_a(\boldsymbol{g})$. (Obviously, this exclusion does not reduce the information used by agent a.) The elements corresponding to F_1 in the first four rows of $\boldsymbol{A}_a$ are set to zeros and the fifth row (corresponding to F_1 itself) in $\boldsymbol{A}_a$ is already known from the model specification. Symmetrically, we exclude F_2 from the regressors used to calculate $\boldsymbol{\Gamma}_b(\boldsymbol{g})$.

Define $\boldsymbol{\Gamma}(\boldsymbol{g}) = [\boldsymbol{\Gamma}_a(\boldsymbol{g})', \boldsymbol{\Gamma}_b(\boldsymbol{g})']'$. The algebraic equation system $\boldsymbol{g} = \boldsymbol{\Gamma}(\boldsymbol{g})$ can yield a closed form solution for $\boldsymbol{g}$ which characterizes the rational expectations equilibrium with imperfect and differential information.

5.4. Noisy asset supplies and information revelation

An important ingredient in our asset pricing model is noisy asset supplies. In the finance literature, noise is often interpreted as exogenous random supply

(Hellwig, 1980; Diamond and Verrecchia, 1981; Admati, 1985) or trading of liquidity/noise traders (Kyle, 1985, 1989; De Long et al., 1990; Campbell and Kyle, 1993; Zhou, 1997). The noisy rational expectations framework has now been widely and successfully used in various asset pricing models (Lang et al., 1992).

Theoretically, in a perfect securities market without noise, if asymmetric information is the only motivation for trading, then an agent reveals his or her information to the market by his or her willingness to trade. Hence, information is fully revealed in equilibrium and no trade actually occurs as new information comes in. This is the so-called 'no-trading theorem' noticed by Grossman (1981) and Milgrom and Stokey (1982). When noise trading is present, however, private information may not be fully revealed since agents may not know if the trading is driven by noisy asset supplies or by private information. (See, e.g., Hellwig, 1980; Diamond and Verrecchia, 1981; Kyle, 1985, 1989; Wang, 1993; Zhou, 1997).

Sargent (1991) points out that the full-revelation property of a limited-information REE is related to the dimension of the space of (price) signals relative to the dimension of the private information set. Sargent's model has two price signals and two privately observed information variables. The equilibrium of his model turns out to be a pooling equilibrium or a fully revealing equilibrium. There are also two price signals and two privately observed information variables in our model, but the equilibrium of our model is not a pooling equilibrium (see the next section for numerical illustrations). Private information is not fully revealed and differently informed agents hold different portfolios.

The difference between the information revelation properties of Sargent's model and our model reflects the effects of noisy asset supplies on the information extraction in equilibrium. In Sargent's model, prices are determined by publicly observable state variables and private information variables. If the dimension of price signals is the same as that of the private information set, observing price signals will provide enough information for one class of economic agents to figure out the private information of the other class. In our model, prices are determined not only by publicly observable state variables and private information variables, but also by noisy asset supplies, as shown in Eq. (51). Noisy supplies can affect asset prices because they affect the market clearing condition. They can also affect equilibrium asset holdings of rational agents. Since price movements reflect both private information variables and unobservable noisy asset supplies in a noisy market, an agent who wants to use price signals to figure out the private information of other agents must figure out the noisy asset supplies simultaneously. One cannot completely determine the private information of other agents if the dimension of price signals is less than the dimension of noisy asset supplies plus the dimension of private information variables. In our model, there are two price signals. These signals are not sufficient to fully reveal noisy asset supplies (two variables) and private information (two variables).

If rational agents can observe noisy asset supplies in our model, the private information on asset fundamentals will be fully revealed by price signals. A pooling equilibrium or a fully revealing equilibrium is then established. This equilibrium is equivalent to the perfect information equilibrium discussed earlier since agents can know the current (and past) values of all state variables by observing asset prices.

6. Numerical examples

This section provides some numerical simulations. The major purpose of these simulations is to give some basic intuition for the effects of differential information on the stock market. We will discuss a number of interesting results in this section.

6.1. Specifying parameter values

To highlight the effects of differential information on the stock market, and to make comparison and exposition more convenient, we will consider two symmetric assets with the same moments. Our major purpose is to show some basic intuition for our theory, so no effort will be made to match the parameters with historical data.

$$r = 0.05, \tag{63}$$

$$\boldsymbol{a}_F = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}, \tag{64}$$

$$\boldsymbol{a}_N = \boldsymbol{0}, \tag{65}$$

$$\boldsymbol{\Sigma}_D = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \tag{66}$$

$$\boldsymbol{\Sigma}_F = \begin{bmatrix} 1.0 & 0.2 \\ 0.2 & 1.0 \end{bmatrix}, \tag{67}$$

$$\boldsymbol{\Sigma}_N = \begin{bmatrix} 0.04 & 0.0 \\ 0.0 & 0.04 \end{bmatrix}. \tag{68}$$

6.2. Numerical results for the perfect information model

As a benchmark case, we give the numerical results for the perfect information model first. With the parameters specified above, we obtain

$$\boldsymbol{\Sigma}_{\Pi} = \begin{bmatrix} 6.0205 & 1.2944 \\ 1.2944 & 6.0205 \end{bmatrix}. \tag{69}$$

This implies that the required constant price discount is $p_1 = p_2 = -146.3$ and that the price function is

$$\boldsymbol{P}_t = -\begin{bmatrix} 146.30 \\ 146.30 \end{bmatrix} + \boldsymbol{V}_t - \begin{bmatrix} 5.734 & 1.233 \\ 1.233 & 5.734 \end{bmatrix} \boldsymbol{N}_t, \tag{70}$$

where $\boldsymbol{V}_t = 0.909\boldsymbol{F}_t$ is the vector of stock fundamentals.

6.3. Numerical results for the differential information model

In the perfect information model, every rational economic agent shares the same information and holds the same portfolio – the noisy market portfolio. The situation changes dramatically in the differential information setup. Table 1 shows an example about the information extraction with $k = 0.5$. It only gives the computational results for agent a because the results for agent b are completely symmetric when $k = 1 - k = 0.5$.

Matrices $\boldsymbol{A}_a$ and $\boldsymbol{B}_a$ completely characterize the rationally perceived law of motion of state variables in the equilibrium. They demonstrate that economic agents forecast the future using both the historical values of observable variables and their own past forecast errors. Matrix $\boldsymbol{\Sigma}_{Xa}$ shows the variances (covariances) of agent a's forecast errors. As we expect, since agent a has more information on the fundamental value of stock 1, the next period's dividend and price of stock 1 are predicted more precisely by a. We can see from Table 1 that the variance of agent a's filtration error on stock 1 is 2.38 while the variance of the filtration error on stock 2 is 2.43.

The moments of excess returns $\boldsymbol{\Pi}$ play a critical role in asset prices and portfolio choices. Table 2 reports these moments. The table contains some interesting findings. First, if we compare the table with $\boldsymbol{\Sigma}_{\Pi}$ under the perfect information equilibrium, we find that the variances here are always greater than their perfect information counterparts, even for the stock for which an agent can observe its fundamentals. This is because the filtration errors of agents who do not observe the fundamentals of a stock will affect the price of that stock and make it more volatile. Stock prices are therefore more volatile in the differential information model than in the perfect information model. Second,

Table 1
ARMA information extraction ($k = 0.5$)

	$\tilde{P}_1$	$\tilde{P}_2$	D_1	D_2	F_1
	0.076	0.009	0.337	0.003	0.000
	0.000	0.023	0.010	0.336	0.000
$\boldsymbol{A}_a$	0.000	0.000	0.500	0.000	0.000
	− 0.006	0.048	0.010	0.434	0.000
	0.000	0.000	0.000	0.000	0.500
	− 0.040	0.003	− 0.309	− 0.001	0.347
	− 0.005	0.007	− 0.012	− 0.134	0.052
$\boldsymbol{B}_a$	0.000	0.000	− 0.500	0.000	0.500
	− 0.014	0.017	− 0.003	− 0.159	0.049
	0.000	0.000	0.000	0.000	0.000
	2.384	0.912	0.961	0.203	0.709
	0.912	2.429	0.192	1.054	0.189
$\boldsymbol{\Sigma}_{Xa}$	0.961	0.192	2.000	0.200	1.000
	0.202	1.054	0.200	2.127	0.200
	0.709	0.189	1.000	0.200	1.000

the population structure affects the signal conveyed by prices. The more people who share the information about a stock, the more the information will be conveyed by stock prices to the people who cannot access it. For example, since agent a has better information about stock 1, it is not surprising to see that, for the most cases in Table 2, $\boldsymbol{\Sigma}_{\Pi a}[1,1] < \boldsymbol{\Sigma}_{\Pi a}[2,2]$. However, when k, the proportion of class a agents, is small, stock 1 has a larger return variance conditional on agent a's information even though a knows its fundamentals better. This is because when the proportion of agents who have better information on the fundamentals of stock 1 is very small ($k \ll (1-k)$), the information about stock 1 revealed by the market will be much less than that about stock 2. As a result, the price of stock 1 is more volatile than that of stock 2 and is harder to predict even for an agent who knows its fundamentals better. This finding tells us that the information distribution among agents has a significant impact on the information efficiency of the stock market. An agent who gains more knowledge about a stock not only improves his or her own information regarding this stock but also improves the information of other agents who do not have this knowledge.

These effects can also be seen from the required price discount vector $\boldsymbol{p}$ and changes in portfolio choices of different agents. Fig. 1 shows clearly that with more and more people getting better information about stock 1 (increase in k), the magnitude (absolute value) of p_1 declines and correspondingly, the magnitude of p_2 rises. Here p_1 and p_2 are unconditional expectations of risk premia per share to stock 1 and stock 2 respectively, as defined before. This finding suggests that changes in information efficiency (associated with the population structure) can affect investment risks and required risk premia.

Comparing $\boldsymbol{p}$ in Fig. 1, and in the perfect information model reported in the previous subsection, we can see clearly that the differential information model gives higher risk premia no matter what the population structure is.

Table 2
Volatility of Π

k	$1-k$	$\Sigma_{\Pi a}$		$\Sigma_{\Pi b}$	
0.000	1.000	6.527	1.504	6.934	1.506
		1.504	6.482	1.506	6.134
0.100	0.900	6.487	1.503	6.863	1.506
		1.503	6.533	1.506	6.158
0.300	0.700	6.373	1.506	6.819	1.506
		1.506	6.590	1.506	6.204
0.500	0.500	6.279	1.506	6.690	1.506
		1.506	6.690	1.506	6.279

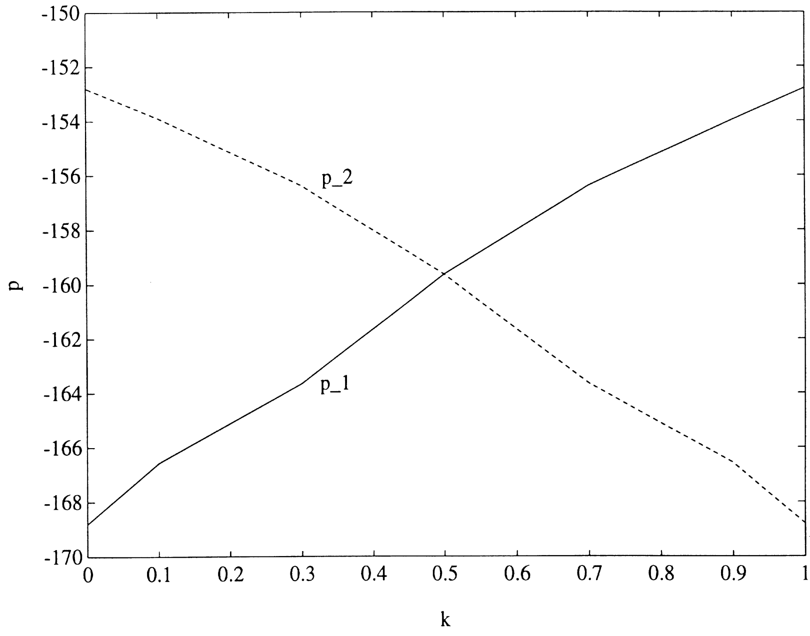

Fig. 1. $\boldsymbol{p}$ vs. k.

Generally speaking, in the steady state, a rational agent tends to hold more shares of stocks with which he or she is more familiar, but at the same time, in order to diversify the portfolio, some shares of stocks for which there is less information about are also held. The portfolio choice depends on the information heterogeneity across the people. Figs. 2 and 3 show the unconditional expectations of the stockholdings of agent a and agent b. We can see that since the increase in the population of class a reduces the risk premium of stock 1 and reveals more information about this stock to class b agents, the private information about stock 1 tends to become less valuable when k becomes larger. Agents have to adjust their portfolios to respond these changes. With the decline in the information advantage regarding stock 1, class a agents will reduce their holdings of that stock. At the limit of $k \to 1$, they hold the market portfolio eventually.

How far could a rational agent go away from the market portfolio in a noisy asymmetric information market? The answer is that it depends on the extent of information asymmetry. Numerical results (not reported here) show that when we increase the variances of the noise terms, the equity portfolio of an agent will typically become less diversified since less information will be transmitted by stock prices, and information asymmetry becomes more important. When the

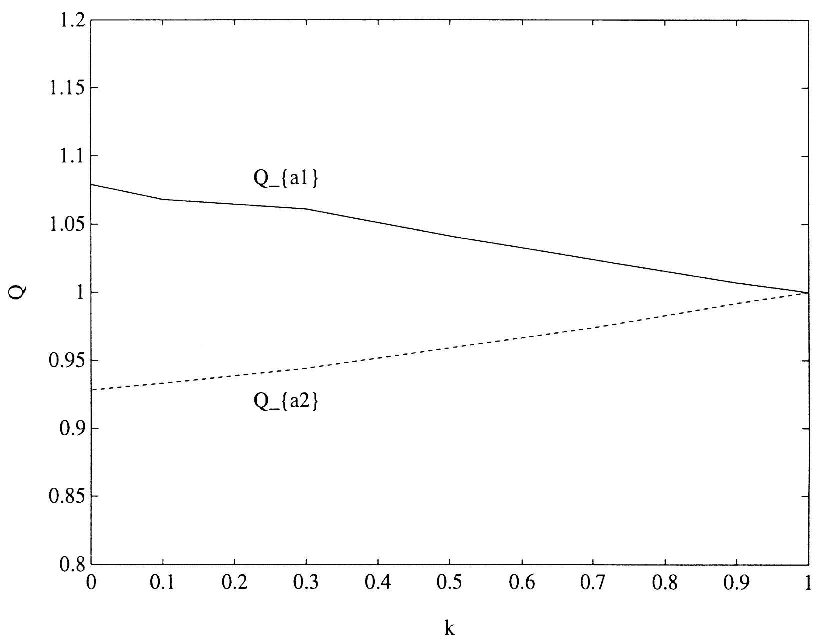

Fig. 2. $\boldsymbol{Q}_a$ vs. k.

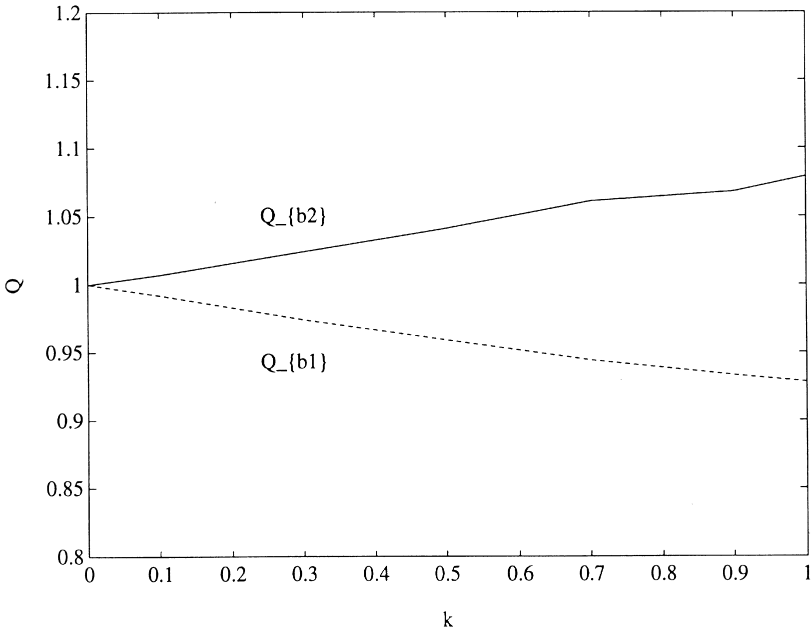

Fig. 3. $\boldsymbol{Q}_b$ vs. k.

variances of the noise terms (especially σ^2_{2N}) become large enough, we can find, for example, $Q_{a1} \gg 1$ and Q_{a2} gets very close to 0. If we interpret each asset as the portfolio of a specific country and assume that agents have more information on domestic financial markets, this result may help to explain the 'home bias puzzle' in international portfolio choices.

The differences in information and portfolio choices may affect the welfare of economic agents significantly. Table 3 presents the unconditional means and the unconditional variances of excess returns obtained by agents from zero wealth (buying stocks by selling the same amount of riskfree bonds). In this table, e_{ra} and e_{rb} are the unconditional means of excess returns generated by portfolios held by agents in class a and class b respectively; σ_{ra} and σ_{rb} are the corresponding unconditional standard deviations. The larger group, usually earn lower portfolio returns due to stronger competition inside the group and more private information that is transmitted by prices to other agents. That is why the stock market information shared by a smaller number of agents is more valuable than the same information shared by a larger number of agents. It can also explain why a small number of insiders often make very high extra profits on a stock. Fig. 4 shows the mean utilities generated by zero initial net wealth for both classes of agents. Agents in class a enjoy a much higher mean utility

Table 3
Portfolio returns

k	$1-k$	e_{ra}	σ_{ra}	e_{rb}	σ_{rb}
0.000	1.000	16.756	4.025	16.603	4.010
0.100	0.900	16.662	4.009	16.552	4.003
0.300	0.700	16.605	4.008	16.521	3.997
0.500	0.500	16.513	3.996	16.513	3.996
0.700	0.300	16.521	3.997	16.605	4.008
0.900	0.100	16.552	4.003	16.662	4.009
1.000	0.000	16.603	4.010	16.756	4.025

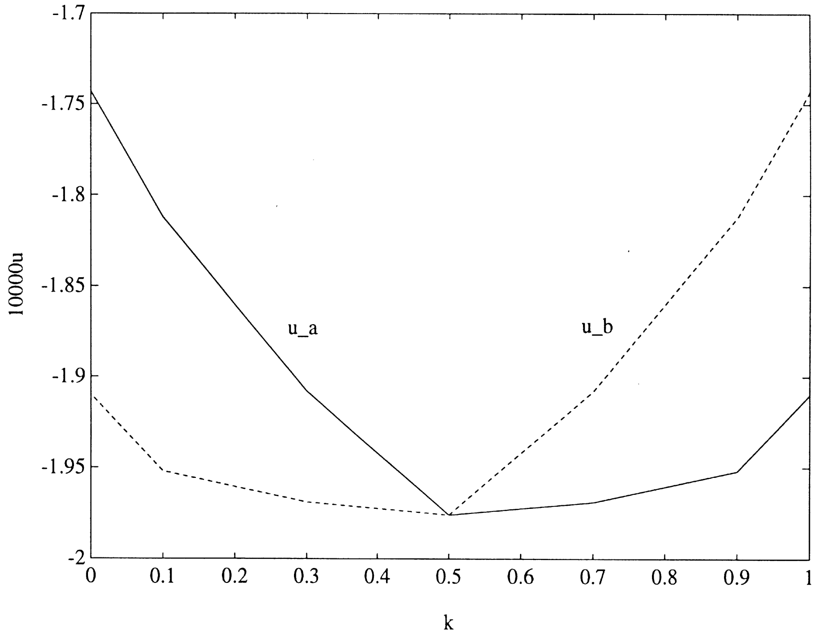

Fig. 4. Utility level.

level when a has a very small population, or k is close to zero. This finding again shows that the private information shared by a small group is very valuable. From Fig. 4 (and also Table 3), we can see that when the population in class a is large enough, a's utility u_a, can also be an increasing function of k. This is because when k rises, the population in class b declines. Though class a agents will lose their super information about stock 1 to other people when k gets larger, they will suffer less disadvantages due to ignorance about stock 2 because fewer people can take advantage of their ignorance. In summary, the population structure k has two effects on agents' welfare. For agents in class a, when k is pretty small, the first effect (their information priority in stock 1) dominates and u_a is a decreasing function of k; when k is big enough, the second effect (their information disadvantage in stock 2) dominates and their utilities will have a positive slope after k exceeds a certain value. Utilities of class b agents are symmetric to those of class a agents.

The welfare implication of Fig. 4 is intriguing. All agents can be better off when k is either large (close to 1) or small (close to 0). Therefore, we would rather have agents being informed about only one stock than have half of the agents being informed about half of the stocks.

7. Conclusions

This paper considers asset pricing and portfolio choices in a multi-asset securities market with a fairly general information setup. The information is heterogeneous and not completely ranked.

The differential information, multi-asset models have a number of important implications which cannot be found in a single-asset model or a homogeneous information model. The information structure has a significant impact on asset prices and portfolio choices. An agent tends to hold more stocks which he or she knows better, but may still need to hold some other stocks to diversify the portfolio. The diversification depends upon the importance of information asymmetry. An agent may hold a portfolio very different from the fully diversified one if the market is very noisy. If we consider instantaneous portfolio choices, we can find that sometimes an agent may hold a smaller proportion of stocks that are better known. This situation happens when an agent's private information about a stock shows that this stock is overpriced due to the filtration errors of other agents.

The stock prices convey a part of private information to the public. The revealed information in turn affects stock prices themselves. Since the risk premium is inversely proportional to the information precision regarding the stock return, the more information that is transmitted, the smaller is the equity premium that is required. The paper shows that high quality insider information shared only by a small number of agents is often considerably valuable. Agents with this kind of insider information can make large extra profits and become substantially better off. It also shows that the information structure of the stock market has very important welfare implications for the whole economic society, i.e., the diversity of information distribution among agents may reduce the welfare of all agents.

The model presented here is promising since it explains a number of empirical findings regarding the equity premium, market volatility, imperfect diversification, and home bias.

The model in this paper investigates an information structure where each agent has superior information about some assets in the economy but inferior information about others. Therefore no one has better information in every aspect. Another interesting information premise is that some agents have better information about all assets. A non-revealing equilibrium can also be established in this setup. In the equilibrium, less informed agents will face an adverse selection problem in the sense that they may buy overvalued assets from, and sell undervalued assets to, better informed agents. This information structure has been investigated by Zhou (1997).

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